Cosmology

The Discovery of the Universe

• The Great Debate was resolved in 1924 by a 35 year old Edwin Hubble, who used the 100″ Hooker Telescope on Mount Wilson to observe Cepheids in nearby spiral nebulae, proving they were galaxies unto themselves.

• The discovery of the galaxies dispelled the idea that the Milky Way was the entire Universe, and work on *cosmography* — the mapping of the Cosmos — began in earnest.

• In 1929, using observations of his own, combined with other data from Milton Humason (who also worked at Mount Wilson), and Vesto Slipher (who worked at Lowell Observatory), Hubble published one of the most important papers in Cosmology — *A Relation Between Distance and Radial Velocity Among Extra-galactic Nebulae* (PNAS **15**, 168 [1929], available online at `www.pnas.org/content/15/3/168.full.pdf+html`)

• Part of sifting through vast amounts of astronomical data, particularly when you are trying to learn what the principles governing the dynamics and evolution of astrophysical systems, is looking for and discovering correlations between different observable physical properties.

• Hubble, using the new Cepheid variable technique, was able to measure the distances to 46 galaxies. He was also able to measure the speed of the galaxies relative to our own using Doppler shifts. When he compared these two seemingly disparate quantities, he made a great discovery: *on large scales, all the galaxies are moving away from us; the farther away they are, the faster they are receding.*

• Hubble's conclusions are illustrated in Figure 1 of his famous paper, shown below

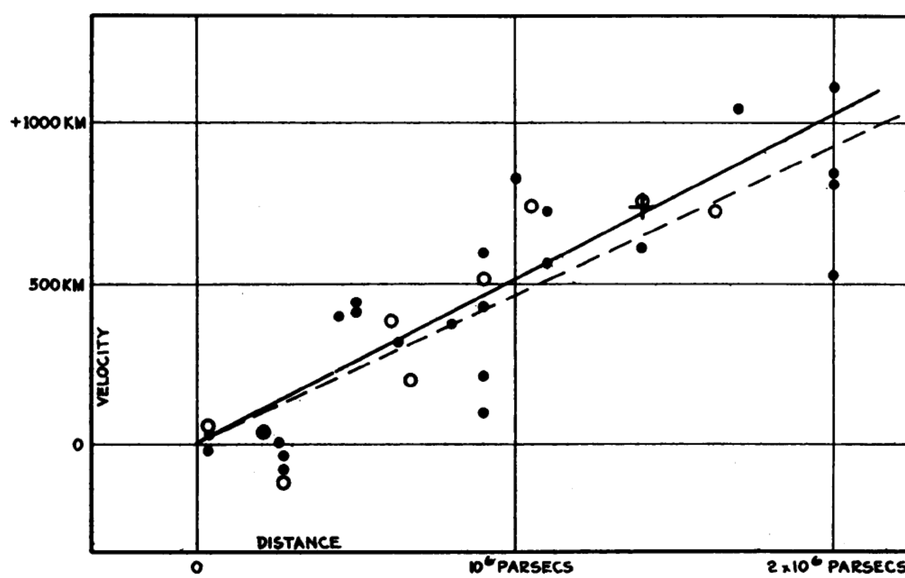


FIGURE 1

• Hubble did what any good physics student would do, and made a linear fit to the data (the line in the figure). The equation describing that line is now known as the *Hubble Law*:

$$v = H_o d$$

• Here v is the *recessional velocity*, d is the *proper distance*, and the slope of this line, H_o, is called the *Hubble constant*.

• So what does this mean? The interpretation is that *the Universe is expanding*. Every galaxy is racing away from every other galaxy because the space between them is expanding!

• An often used analogy is to pretend you live in a two-dimensional Universe. One good model to think about this is the surface of a balloon.

- ▷ Pick up a balloon, and inflate it slightly; draw several galaxies on its surface and measure the distance between each galaxy.
- ▷ Now inflate the balloon a bit more. What happens? The rubber *stretches*; in the language of the Cosmos, *the space between the galaxies is expanding*.
- ▷ Now measure the distance between all the galaxies — *every galaxy got farther away from every other galaxy*. The reason? The space between every galaxy expanded.

• A similar model can be created in three dimensions using a loaf of raisin bread — space is represented by the dough, and the galaxies by raisins. Before you bake the loaf, measure the distances between raisins in a slice through your dough. After baking, measure the distances again, and you will find that the distance between every raisin increased.

Cinnamon Raisin Bread Recipe

- 1-1/2 cups milk
- 1 cup warm water
- 2 (0.25 oz) packages yeast
- 3 eggs
- 1/2 cup white sugar
- 1 tsp salt
- 1/2 cup margarine
- 1 cup raisins
- 8 cups flour
- 2 tbsp milk
- 3/4 cup white sugar
- 2 tbsp cinnamon
- 2 tbsp butter, melted

► Warm the milk in a small saucepan until it bubbles, then remov e from heat. Let cool until lukewarm.
► Dissolve yeast in warm water, and set aside until yeast is fro thy. Mix in eggs, sugar, butter or margarine, salt, and raisins. Stir in cooled milk. Add the flour gradually to make a stiff dough.
► Knead dough on a lightly floured surface for a few minutes. Place in a large, greased, mixing bowl, and turn to grease the surface of the dough. Cover with a damp cloth. Allow to rise until doubled.
► Roll out on a lightly floured surface into a large rectangle 1/2 inch thick. Moisten dough with 2 tablespoons milk. Mix together 3/4 cup sugar and 2 tablespoons cinnamon, and sprinkle mixture on top of the moistened dough. Roll up tightly; the roll should be about 3 inches in diameter. Cut into thirds, and tuck under ends. Place loaves into well greased 9 x 5 inch pans. Lightly grease tops ofl oaves. Let rise again for 1 hour.
► Bake at 350° F for 45 minutes, or until loaves are lightly browned and sound hollow when knocked. Remove loaves from pans, and brush with melted butter or margarine. Let cool before slicing.

• The *cosmological redshift* is not a Doppler shift because it is not due to the relative motion of the source and the observer; it is due to the *expansion of space*. However, the observations of galaxies have traditionally been analyzed in the context of the Doppler shift, hence the terminology *recessional velocity*.

• Because of this connection, it is convenient to express the interpreted observation, the velocity v, in terms of the actual observed quantity, the redshift z. We can use the definition of the redshift and the Doppler shift to do this:

$$z = \frac{\lambda_{obs} - \lambda_{emit}}{\lambda_{emit}} \qquad \rightarrow \qquad 1 + z = \frac{\lambda_{obs}}{\lambda_{emit}}$$

• This is related to the Doppler effect by

$$1 + z = \sqrt{\frac{1 + v/c}{1 - v/c}} \qquad \rightarrow \qquad 1 + z = (1 + v/c)^{1/2}(1 - v/c)^{-1/2}$$

• Applying the binomial expansion to each of the terms on the right hand side gives

$$1 + z \simeq \left(1 + \frac{v}{2c}\right)\left(1 + \frac{v}{2c}\right) = \left(1 + \frac{v}{2c}\right)^2$$

• Applying the binomial expansion *again* gives

$$1 + z \simeq 1 + \frac{v}{c} \qquad \rightarrow \qquad z \simeq \frac{v}{c}$$

• Using this with the Hubble law then yields

$$z = \frac{H_o}{c} d$$

The Hubble Constant

• A great deal of effort has been expended by astronomers to determine the value of the Hubble constant, H_o. One of the primary justifications for building the Hubble Space Telescope was the *HST Key Project*, to determine the most accurate value of the Hubble constant known at that time.

• Early on in the development of observational cosmology, there were two threads of evidence (and theoretical argument) about the value of H_o — the *big* H_o crowd favored values of $H_o \simeq 80$ km/s/Mpc, and the *low* H_o crowd, favoring values of $H_o \simeq 65 km/s/Mpc$. Even today there is a spread in values for the H_o, but we have settled on values in the low 70's. Combining multiple sources of observations together the current best value is

$$H_o = 70.4\ km/s/Mpc$$

• What are with those units? They make sense based on Hubble's graph — the slope is just the Hubble constant. But reducing the units to their fundamental base units, we find *the Hubble constant has units of 1/TIME.*

• For the current value of H_o the *Hubble time* is

$$\tau_H = \frac{1}{H_o} = 4.38 \times 10^{17}\ s = 13.9 \times 10^9\ yr$$

• If the Hubble Law is valid through the entire observable Universe, then the Hubble time is an estimate of the age of the Universe.

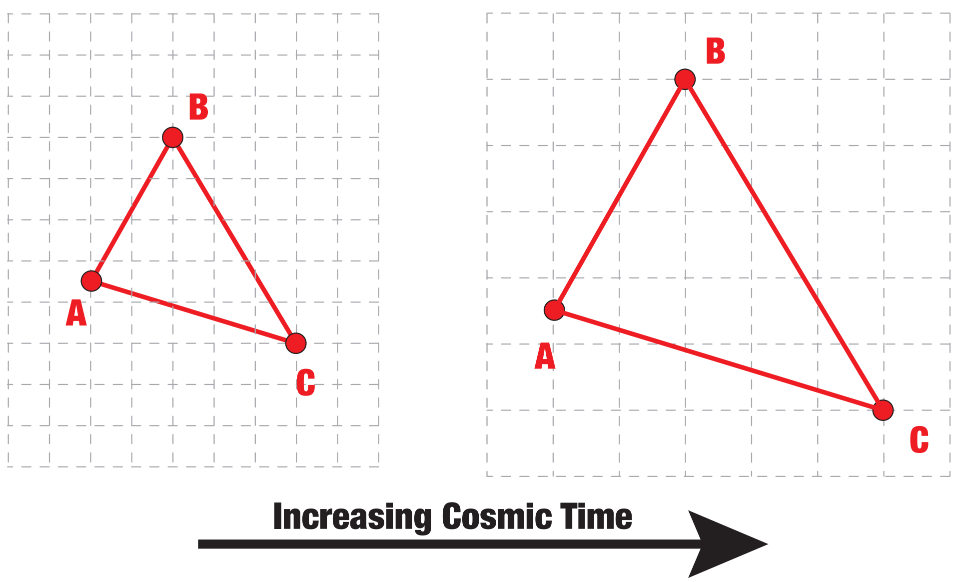
The Expansion of the Cosmos

• The expansion of the Cosmos was predicted and described theoretically before it was discovered. In 1927, Georges Lemaitre used general relativity to derive an equation for the expansion rate, what is now called the Hubble constant.

• We are not going to go through the derivation of the equations governing cosmology, but we will state (without proof) the equations essential to being able to understand the behaviour and evolution of the Cosmos.

• If we want to mathematically describe the expansion of the Cosmos, we have to agree on the quantities we are talking about.

• Astronomers generally think of measuring the Cosmos the way we think of measuring anything — we define *coordinates* used to define locations (in space and time). A common set of coordinates in cosmology are the *comoving coordinates*. As their name suggests, they are rigidly attached to objects in the Cosmos (like galaxies) and expand as the Cosmos expands. This is shown in the figure below.



• The other common distance measure is the *proper distance*; it is the invariant distance between two points in spacetime defined by relativity. In the context of cosmology, it is useful to think of the proper distance as the distance you would measure between two points if you could take a snapshot and freeze the expansion of the Cosmos for the time it takes you to measure the distance of interest.

• At a particular cosmic time t, the proper distance $d_p = d_p(t)$ is related to the comoving distance d_o by a function called the ***scale factor***, $a(t)$ [1]

$$d_p = a(t) \cdot d_o$$

• The scale factor represents the relative expansion of the Cosmos. Hubble characterized the expansion using the constant H_o, so it must be related to the scale factor, and it is! If $t = t_o$ is the value of cosmic time today, then

$$H_o = \left(\frac{\dot{a}}{a}\right)_{t=t_o}$$

The Stuff in the Cosmos

• We have been writing the scale factor as $a(t)$, implicitly suggesting that it can evolve as a function of cosmic time. The equations that govern the expansion are called the *Friedmann equations*, first derived from general relativity by the Russian mathematical physicist, Alexander Friedmann. Sadly, Friedmann died of typhoid fever in 1925 at the age of 37, before the expansion of the Cosmos was observationally discovered.

• The expansion of the Cosmos depends on *what is in the Cosmos*. To address this question, we adopt our intuition and strategy from our understanding of fluid dynamics. As such, the quantities of interest are the *densities* and *pressures* of the stuff filling the Cosmos.

• There are several components to what fills the Cosmos: *matter*, *radiation*, and as we will discover soon, *dark energy*. We write the density and pressure in the Cosmos as

$$\rho(t) = \rho_m(t) + \rho_r(t) + \rho_\Lambda$$
$$p(t) = p_m(t) + p_r(t) + p_\Lambda$$

• We have certain expectations for how each of these quantities evolves with the cosmic expansion. Let's think about densities first. Consider a cube filled with each of the types of stuff. Over some time, imagine that the uniform expansion of the Universe causes the sides of the cube to increase their proper length by a factor of 2 (i.e. a doubling of $a(t)$), so the proper volume increases by a factor of 8.

- ▷ Both ρ_Λ and p_Λ are expected to be independent of time
- ▷ The number density n of particles has decreased by a factor 8 (corresponding to the increase in proper volume). The mass of the particles m has *not* changed, and since $\rho_m = m \cdot n$, it must be the case that the mass density scales as

$$\rho_m \propto 1/a^3$$

[1] Be aware that there is not a universally accepted symbol for the scale factor; $R(t)$ is also common.

▷ Since radiation is massless, the density ρ_r is related to the energy density of the radiation. We have already seen from our study of the Hubble law that the cosmic expansion causes radiation to be *redshifted*; doubling the scale factor redshifts the wavelength $\lambda \to 2\lambda$, which cuts the energy of a bit of radiation in half (i.e. by a factor of $1/a$). We've already seen that the number density decreases by a factor of $1/a^3$, so overall the effect on radiation is

$$\rho_r \propto 1/a^4$$

• Pressure and density are related to one another by an ***equation of state***. In cosmology, we use the form

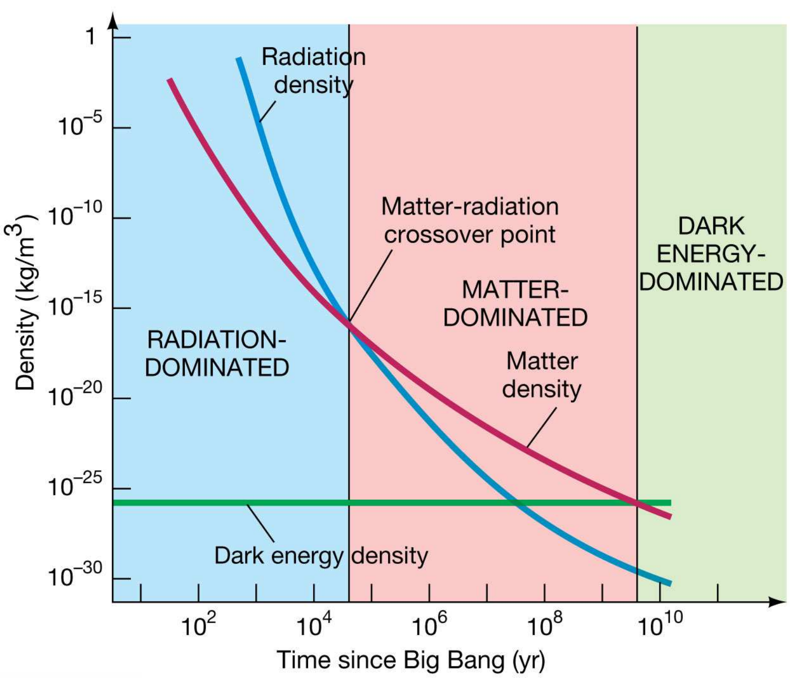
$$p_i = w_i \rho_i c^2$$

where the parameter w is call the equation of state parameter. Every constituent i has a corresponding value of w_i that relates the pressure to the density.

• For the different constituents we have been considering:

$$w_\Lambda = -1 \qquad w_{rad} = \frac{1}{3} \qquad w_m \geq 0$$

• The physics and evolution of the Cosmos at any given moment is governed by which constituents have dominance in terms of their overall contribution to the energy density of the Universe. Using the dependences noted above, the energy density as function of cosmic time can be plotted as below.



• In different eras, different constituents dominate. As the Cosmos expands, the relative importance of one constituent to another changes. Cosmologists believe that we currently live on the far right of this graph, in the vicinity of the cross-over point from matter dominance to dark energy dominance.

• Based on our scaling laws, we can use ratio arguments to write down the time evolution of the densities. If $a_o = a(t_o)$ is the scale factor today, and $\rho_o = \rho(t_o)$ is the density today, then at some other time t then

$$\frac{\rho_m(t)}{\rho_{m,o}} = \left[\frac{a_o}{a(t)}\right]^3 \qquad \rightarrow \qquad \rho_m(t) = \rho_{m,o}\left[\frac{a_o}{a(t)}\right]^3$$

$$\frac{\rho_r(t)}{\rho_{r,o}} = \left[\frac{a_o}{a(t)}\right]^4 \qquad \rightarrow \qquad \rho_r(t) = \rho_{r,o}\left[\frac{a_o}{a(t)}\right]^4$$

• This works great, *as long as you know the scale factor!* As it turns out, the scale factor is governed by a set of evolution equations derived from general relativity known as the *Friedmann Equations*.

The Friedmann Equations

• The Friedmann equations are simply the result of writing down the Einstein Field Equations for a metric known as the *Robertson-Walker* metric.

• For your edification the Robertson-Walker spacetime metric is

$$ds^2 = -c^2dt^2 + a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

• Here the constant k is called the *curvature constant*. It has values $k = \{-1, 0, +1\}$ and defines the curvature of the Cosmos

• We will *not* do the derivation, opting instead to simply state the two Friedmann equations. They are

$$\left[\frac{1}{a}\frac{da}{dt}\right]^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$$

$$\frac{1}{a}\frac{d^2a}{dt^2} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)$$

• The first equation is often called the *energy equation*. With a little algebraic massage it may be rewritten as

$$\frac{1}{2}\left[\frac{da}{dt}\right]^2 - \frac{G}{a}\frac{4\pi a^3}{3}\rho = -\frac{kc^2}{2} = const$$

• This now just looks like the specific kinetic energy + the specific gravitational potential energy is equal to a constant — *looks like conservation of energy!*

• The second equation is often called the *acceleration equation*. A little algebraic massage makes this look like

$$\frac{d^2a}{dt^2} = -\frac{G}{a^2}\frac{4\pi a^3}{3}\left(\rho + \frac{3p}{c^2}\right)$$

• This now looks like the acceleration at the surface of a sphere of radius a with a uniform density $\rho + 3p/c^2$

• Another common practice among cosmologists is to combine the two Friedmann equations. Taking a page out of our fluid books, differentiating the energy equation to express d^2a/dt^2 in the acceleration equation leads to the *fluid equation*

$$\frac{d\rho}{dt} + \left(\rho + \frac{p}{c^2}\right)\frac{3}{a}\frac{da}{dt} = 0$$

Cosmological Models

• As observers, we are very interested in what the structure, history and ultimate fate of the Universe is. The tools of cosmology give us the ability to address such questions, but ultimately we are limited by our observational capabilities. Theory has traditionally played a strong role in cosmology by taking different conditions in the Friedmann equations, and making predictions that can be observationally tested.

• Suppose your take our scaling laws for the densities ρ and the pressures p and put them into the Friedmann Equations. The results are a set of expanded, fundamental equations describing the evolution of the Cosmos as a function of what it is filled with! The fully expanded and simplified equations are

$$\left[\frac{1}{a}\frac{da}{dt}\right]^2 = \frac{8\pi G}{3}\left\{\rho_{m,o}\left[\frac{a_o}{a(t)}\right]^3 + \rho_{r,o}\left[\frac{a_o}{a(t)}\right]^4 + \rho_\Lambda\right\} - \frac{kc^2}{a^2}$$

$$\frac{1}{a}\frac{d^2a}{dt^2} = -\frac{4\pi G}{3}\left\{\rho_{m,o}\left[\frac{a_o}{a(t)}\right]^3 + 2\rho_{r,o}\left[\frac{a_o}{a(t)}\right]^4 - 2\rho_\Lambda\right\}$$

• There is very strong observational evidence that the Cosmos has $k = 0$ (so called *flat cosmologies*). Let's look at three different $k = 0$ models, specified by different choices of ρ. Any combination of densities could be chosen, but the three models here are pure, single component models

Model 1: de Sitter Universe: $k = 0$, $\rho_{m,o} = \rho_{r,o} = 0$ ▶ . . . .

• This is a Universe filled with nothing except *dark energy*. First proposed in 1917 by William de Sitter, it was the first cosmological model to describe an expanding Universe.

• Imposing the density and k values, the energy equation becomes

$$\frac{da}{dt} = a\sqrt{\frac{8\pi G}{3}\rho_\Lambda}$$

• This is a first order differential equation. To integrate it, we require one initial condition (to specify the constant of integration), which is generally taken to be that the value of the scale factor is known today, at time $t = t_o$ such that $a_o = a(t_o)$. The solution then is

$$a(t) = a_o \exp\left[\sqrt{\frac{8\pi G}{3}\rho_\Lambda}(t - t_o)\right]$$

• With a solution to $a(t)$ we can write how the Hubble constant evolves in this cosmology

$$H(t) = \frac{\dot{a}}{a(t)} = \sqrt{\frac{8\pi G}{3}\rho_\Lambda}$$

• This is *constant* if $\rho_\Lambda = const$, so $H(t) = H(t_o) = H_o$ and the solution simplifies to

$$a(t) = a_o \exp\left[H_o(t - t_o)\right]$$

• Since the Cosmos has matter and radiation in it, this is not a good approximation to the current Cosmos. However, if there is a cosmological constant, then eventually its density must dominate the Cosmos, and this model will accurately describe the far future of the Universe.

Model 2: Flat Radiation Universe: $k = 0$, $\rho_{m,o} = \rho_\Lambda = 0$ ► ....

• This is a Universe filled with nothing but radiation. Given our current understanding of the early Universe, this is a good model for the high energy era, just following the Big Bang and inflation.

• As with the de Sitter model, our fundamental evolution equation can be found from the energy equation

$$\frac{da}{dt} = \frac{a_o^2}{a}\sqrt{\frac{8\pi G}{3}\rho_{r,o}}$$

• This first order equation can be integrated assuming that $a_o = a(t_o)$ at the current epoch with result

$$a(t) = a_o\sqrt{2H_o t}$$

where

$$H_o = \sqrt{\frac{8\pi G}{3}\rho_{r,o}}$$

Model 3: Einstein-de Sitter Universe: $k = 0$, $\rho_{r,o} = \rho_\Lambda = 0$ ►.......................

• After the discovery of the expansion of the Universe, Einstein strongly favored a Cosmos with $\rho_\Lambda = 0$, since he had introduced the cosmological constant the make *static* cosmological models. In 1932, Einstein and de Sitter proposed this model (containing only matter); for many decades it was regarded as a good model to describe the observable Universe.

• The model became less favored as observational data became better, and since the 1990's has been disfavored as dark energy observations have caused a resurgence in models with $\rho_\Lambda \neq 0$ that agree better with current data.

• In this model, the energy equation becomes

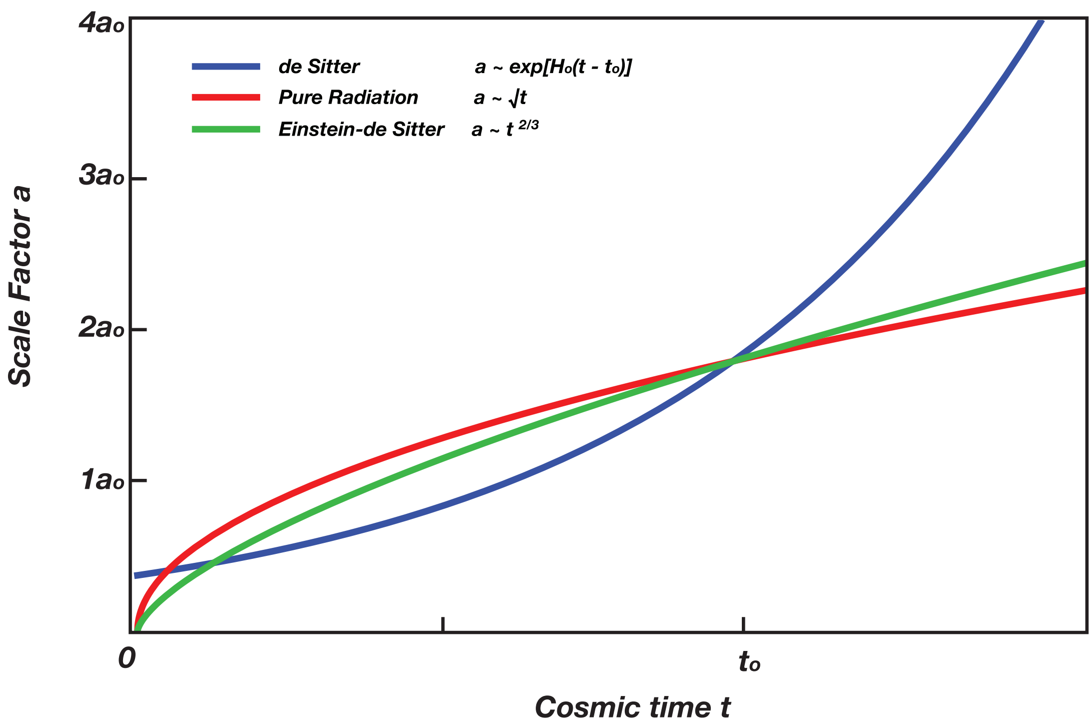
$$\frac{da}{dt} = \frac{a_o^{3/2}}{a^{1/2}} \sqrt{\frac{8\pi G}{3}\rho_{m,o}}$$

• This first order equation can be integrated assuming that $a_o = a(t_o)$ at the current epoch with result

$$a(t) = a_o \left[\frac{3}{2} H_o t\right]^{2/3}$$

where

$$H_o = \sqrt{\frac{8\pi G}{3}\rho_{m,o}}$$



General Cosmological Models

• Generically, the Hubble constant has the same form for each of the three cosmological models we considered above

$$H(t) = \sqrt{\frac{8\pi G}{3}\rho(t)}$$

• This is the generic solution for a $k = 0$ Universe. The value of the cosmic density defined by this relationship is generally called the *critical density* $\rho_c(t)$ and is given by

$$\rho_c(t) = \frac{3H^2(t)}{8\pi G}$$

• The physical interpretation of this density is that it is the boundary value between cosmological models that collapse (*closed* models) and models that expand forever (*open* models). Generically, these models are defined by the value of the curvature constant k: $k < 0$ for open models, and $k > 0$ for closed models.

• To see the connection between the critical density ρ_c and the curvature constant k, look at the energy equation.

$$\left[\frac{1}{a}\frac{da}{dt}\right]^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} \qquad \rightarrow \qquad H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$$

• Writing H^2 in terms of ρ_c yields

$$\frac{8\pi G}{3}(\rho - \rho_c) = \frac{kc^2}{a^2}$$

• This shows that $k = 0$ cosmologies correspond to the case where $\rho = \rho_c$.

• It is common practice to work with the *density parameter*

$$\Omega \equiv \frac{\rho(t)}{\rho_c(t)} \qquad \rightarrow \qquad \Omega_m = \frac{\rho_m}{\rho_c} \quad , \quad \Omega_r = \frac{\rho_r}{\rho_c} \quad , \quad \Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c}$$

• We have only written down the simplest of cosmological models, with single component densities. In more complicated models, cosmologists include more than one density component in the model. This often makes it difficult or impossible to derive analytic results, and numerical methods are used. We will explore the use of numerical methods in cosmology in our upcoming `Maple` exercise.