Binary Stars

Kepler's Laws of Orbital Motion

• Kepler's Three Laws of orbital motion result from the solution to the equation of motion for bodies moving under the influence of a central $1/r^2$ force — gravity.

- ▷ Orbits are ellipses. When one mass dominates (like the Sun in relation to the planets) the dominant mass is at one focus of the ellipse. Generically, the focal point of a two-body system is located at the *barycenter* — the center of mass of the two bodies.
- ▷ As the orbit evolves, the area swept out by the radius vector (pointing from the focus of the orbit to the body) per unit time is a constant: $dA/dt = const$.
- ▷ The square of the orbital period P is proportional to the cube of the semi-major axis a of the orbit: $P^2 \propto a^3$.

Derivation of Kepler III

The basic form of Kepler III can be derived from basic physical principles: the universal law of gravitation and uniform circular motion. Objects in uniform circular motion are kept in circular motion by a *centripetal force*. If they have a speed v and an orbital radius r then the centripetal force is given by

$$F_c = m\frac{v^2}{r}$$

In a bound orbiting system, *gravity is the centripetal force!* It is the *only* force in the system (draw a free body diagram!) so it *must* be the centripetal force. so

$$F_g = G\frac{Mm}{r^2} \qquad \rightarrow \qquad F_g = F_c \qquad \rightarrow \qquad G\frac{Mm}{r^2} = m\frac{v^2}{r}$$

since the system is in uniform circular motion, the speed is uniform and is simply the circumference of the orbit divided by the period P of the orbit:

$$v = \frac{2\pi r}{P} \qquad \rightarrow \qquad v^2 = \frac{4\pi^2 r^2}{P^2}$$

Using this in our force equation gives

$$G\frac{M}{r^2} = \frac{1}{r}\frac{4\pi^2 r^2}{P^2} \qquad \rightarrow \qquad P^2 = \frac{4\pi^2}{GM}r^3$$

Note that in the solar system, $M \sim M_\odot$ in almost all problems, owing to the Sun's large mass. Then in SI units, the constant $4\pi^2/(GM) = 2.9740 \times 10^{-19}\mathrm{s}^2/\mathrm{m}^3$. Converting to units of $\mathrm{yr}^2/\mathrm{AU}^3$ then $4\pi^2/(GM) = 0.9998\mathrm{yr}^2/\mathrm{AU}^3 \simeq 1.000\mathrm{yr}^2/\mathrm{AU}^3$. For this reason, in the solar system Kepler III is often written as $P^2 \equiv a^3$, in units of yr and AU.

Rosetta Stone: Orbital Mumbo Jumbo ▶ ....

- **a = semi-major axis.** The *major axis* is the long axis of the ellipse. The semi-major axis is 1/2 this length.
- **b = semi-minor axis.** The *minor axis* is the short axis of the ellipse. The semi-minor axis is 1/2 this length.
- **e = eccentricity.** The eccentricity characterizes the deviation of the ellipse from circular; when $e = 0$ the ellipse is a circle, and when $e = 1$ the ellipse is a parabola. The eccentricity is defined in terms of the semi-major and semi-minor axes as

$$e = \sqrt{1 - (b/a)^2}$$

- **f = focus.** The distance from the geometric center of the ellipse (where the semi-major and semi-minor axes cross) to either focus is

$$f = ae$$

- ℓ = **semi-latus rectum.** The distance from the focus to the ellipse, measured along a line parallel to the semi-minor axis, and has length

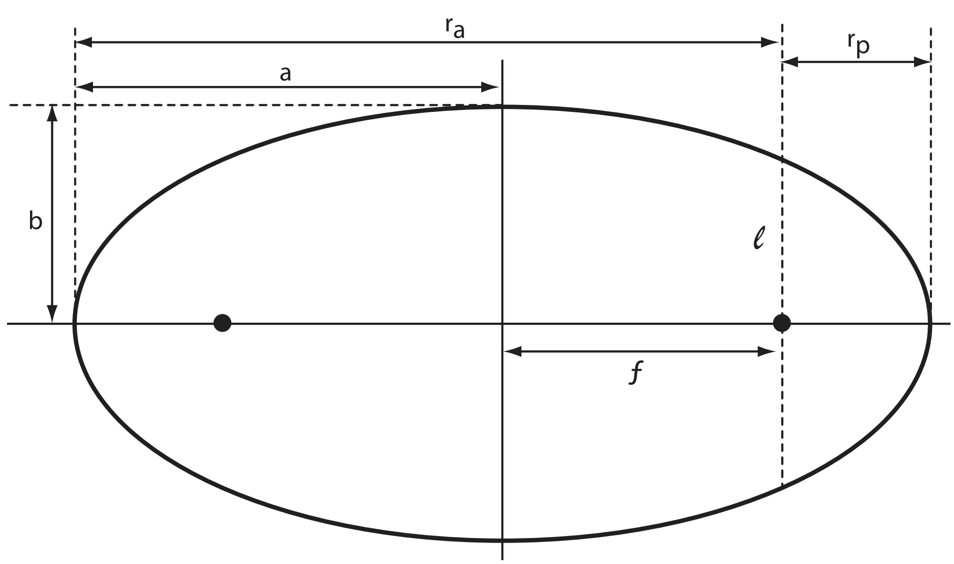
$$\ell = b^2/a$$

- r_p = **periapsis.** The periapsis is the distance from the focus to the nearest point of approach of the ellipse; this will be along the semi-major axis and is equal to

$$r_p = a(1 - e)$$

- r_a = **apoapsis.** The apoapsis is the distance from the focus to the farthest point of approach of the ellipse; this will be along the semi-major axis and is equal to

$$r_a = a(1 + e)$$



Basic Geometric Definitions ▶ . . . .
The game of orbits is always about locating the positions of the masses. For planar orbits (the usual situation we encounter in most astrophysical applications) one can think of the position of the mass m_i in terms of the Cartesian coordinates $\{x_i, y_i\}$, or in terms of some polar coordinates $\{r_i, \theta_i\}$. The value of the components of these location vectors generically depends on the coordinates used to describe them. The most common coordinates used are called *barycentric coordinates*, with the origin located at the focus between the two bodies.

▷ **The Shape Equation**. The shape equation gives the distance of the orbiting body (*"particle"*) from the focus of the orbit as a function of polar angle θ. It can be expressed in various ways depending on the parameters you find most convenient to describe the orbit.

$$r = \frac{a(1-e^2)}{1+e\cos\theta} \qquad \rightarrow \qquad r = \frac{r_p(1+e)}{1+e\cos\theta} \qquad \rightarrow \qquad r = \frac{r_a(1-e)}{1+e\cos\theta}$$

▷ **The Anomaly**. Astronomers refer to the angular position of the body as the *anomaly*. There are three different anomalies of interest.

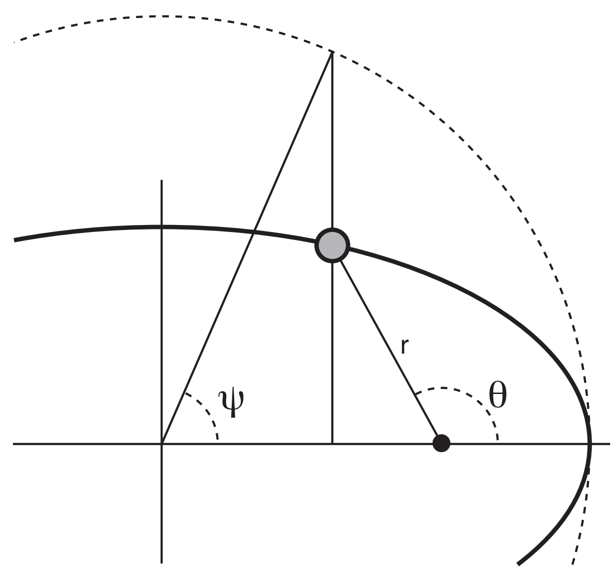
• θ = **true anomaly.** This is the polar angle θ measured in barycentric coordinates.

• $\mathcal{M}$ = **mean anomaly.** This is the phase of the orbit expressed in terms of the time t since the particle last passed a reference point, generally taken to be $\theta = 0$

$$\mathcal{M} = \frac{2\pi}{P} t$$

Note that for *circular orbits*, $\theta = \mathcal{M}$.

• ψ = **eccentric anomaly.** This is a geometrically defined angle measured from the center of the ellipse to a point on a circumferential circle with radius equal to the semi-major axis of the ellipse. The point on the circle is geometrically located by drawing a perpendicular line from the semi-major axis of the ellipse through the location of the particle. The eccentric anomaly is important for locating the position of the particle as a function of time (as we shall see shortly).



Types of Binary Stars

• The taxonomy of binary stars is based largely on the observational properties by which they are identified.

▷ ***Optical doubles.*** These are stars that are not associated with one another other than the fact that they are visible along the same line of sight from Earth; they may be at hugely disparate distances. An excellent example of this is Mizar and Alcor, in the handle of the Big Dipper.

- ▷ ***Visual binary.*** This is a binary where the two components can be resolved in a telescope. If the system is observed for long enough, the entire orbit can be observed. An excellent example is γ Virgins (Porrima), with an orbital period of 168.9 years.
- ▷ ***Astrometric binary.*** This is a binary where one component is much more luminous than the other. The binary nature is detected from the bright component wobbling back and forth under the influence of its companion (*reflex motion*). Sirius was initially discovered to be binary through the astrometric technique, though today even amateur instruments are capable of discerning Sirius B. δ Aquilae, the central star in Aquila, is another well known astrometric binary with a period of 3.4 years.
- ▷ ***Eclipsing binary.*** Some binary orbits are oriented such that they are visible from the Earth edge on. When this occurs, the component stars periodically pass in front of one another from our perspective, causing the light from the system to dip up and down. A famous example of this type is β Persei (Algol) with a period of 2d 20h 49m.
- ▷ ***Spectrum binary.*** These systems are unresolved, but spectroscopic analysis shows there are two super-imposed spectra on one another. There is no observed Doppler shift; the systems may have periods so long as to make the Doppler signal unobservable, or they may be observed down their polar axis.
- ▷ ***Spectroscopic binary.*** These systems are also unresolved and identified by their spectra, but the spectra periodically undergo red and blue shifts. If only one unique Doppler shifted spectrum is observed, these are called *single line binaries*. If two unique spectra are observed, these are called *double line binaries*. In double line binaries, when one spectrum is red-shifted the other spectrum should be blue-shifted.

Mass Determination with Binary Stars

• One of the most important reasons to study binary stars is they can be used to probe the masses of stars. Consider a circular binary. The two components m_1 and m_2 orbit a common center of mass at distances a_1 and a_2 from the center of mass.

$$m_1 a_1 = m_2 a_2 \qquad \rightarrow \qquad \frac{m_1}{m_2} = \frac{a_2}{a_1}$$

• The components in a circular binary are in uniform circular motion, so the component speeds may be expressed in terms of the orbital radii a_i and the orbital period P.

$$P = \frac{2\pi a_1}{v_1} = \frac{2\pi a_2}{v_2} \qquad \rightarrow \qquad \frac{v_1}{v_2} = \frac{a_1}{a_2} = \frac{m_2}{m_1}$$

• In general, a binary will be observed to be inclined to the line of sight be the inclination angle i, the angle between the line of sight and the angular momentum vector of the binary orbit. The angle represents the angle between the plane of the sky and the plane of the orbit. If a circular orbit of radius r is viewed at an angle i, then it will appear as an ellipse with semi-major axis $a = r$ and a semi-minor axis $b = r \cdot \cos i$.

• In a circular binary with orbital speed v inclined by angle i, the component of the velocity along the line of sight (the *radial velocities* v_r, that can be measured via Doppler shifts) has a value $v_{ir} = v_i \cdot \sin i$. Going back to our mass-velocity ratio

$$\frac{m_1}{m_2} = \frac{v_2}{v_1} = \frac{v_{2r}/\sin i}{v_{1r}/\sin i} = \frac{v_{2r}}{v_{1r}}$$

This tells us that measuring the radial velocities will determine the mass ratio in the binary.

• The sum of the masses may be found from the general form of Kepler III:

$$m_1 + m_2 = \frac{4\pi^2}{GP^2}(a_1 + a_2)^3$$

• This may be expressed in terms of the orbital speeds by writing

$$v_i = \frac{2\pi a_i}{P} \qquad \rightarrow \qquad a_i = \frac{P}{2\pi} v_i \qquad \rightarrow \qquad (a_1 + a_2) = \frac{P}{2\pi}(v_1 + v_2) = \frac{P}{2\pi}\frac{(v_{1r} + v_{2r})}{\sin i}$$

• Combining these yields

$$m_1 + m_2 = \frac{P}{2\pi G}\frac{(v_{1r} + v_{2r})^3}{\sin^3 i} \qquad \rightarrow \qquad (m_1 + m_2)\sin^3 i = \frac{P}{2\pi G}(v_{1r} + v_{2r})^3$$

• This is known as *mass-inclination degeneracy* – the mass in a binary may not be determined from the basic observables (P and v_r) without also knowing the inclination of the system.

The Mass Function ▶ . . . .

• In single line binaries, only one of the components is observed, so you only know v_{1r}. From our mass velocity ratio then

$$\frac{m_1}{m_2} = \frac{v_{2r}}{v_{1r}} \qquad \rightarrow \qquad v_{2r} = v_{1r}\frac{m_1}{m_2}$$

• Using this to eliminate v_{2r} is the mass-inclination formula yields

$$(m_1 + m_2)\sin^3 i = \frac{P}{2\pi G}(v_{1r} + v_{2r})^3 \qquad \rightarrow \qquad (m_1 + m_2)\sin^3 i = \frac{P}{2\pi G}v_{1r}^3(1 + \frac{m_1}{m_2})^3$$

$$\rightarrow \qquad (m_1 + m_2)\sin^3 i = \frac{P}{2\pi G} \cdot \frac{v_{1r}^3}{m_2^3}(m_2 + m_1)^3$$

$$\rightarrow \qquad \frac{m_2^3}{(m_1 + m_2)^2}\sin^3 i = \frac{P}{2\pi G}v_{1r}^3$$

• This is known as the *mass-function*. It does not provide useful information on the mass or mass ratio of the system unless the inclination i or some bound on one of the mass components is known. Either m_1 or i is known, this provides a useful lower bound on m_2 since the LHS is *always* less than m_2.

The Kepler Equation

• As a diligent astrophysicist, you time working with orbits will often be spent generating an *ephemeris* — a tabulated list of orbital positions as a function of time.

• Kepler II gives you a way to approach this problem, though it does not give you exactly the most useful result

• Recall that Kepler II says that $dA/dt = const$. This means that the entire area of the ellipse divided by the orbital period P is equal to this constant

$$\frac{dA}{dt} = const = \frac{\pi ab}{P} \qquad \rightarrow \qquad \frac{\pi ab}{P} \cdot t = \int dA$$

• Since the shape equation, $r = r(\theta)$ traces out the ellipse, the area integral can be expressed in polar coordinates, where r is given by the shape equation

$$\frac{\pi ab}{P} \cdot t = \int dA = \int r dr d\theta' = \frac{1}{2} \int_0^\theta r^2 d\theta' = \frac{r_p^2 (1+e)^2}{2} \int_0^\theta \frac{d\theta'}{(1 + e\cos\theta')^2}$$

• Leaving the integration to the interested reader (it is done in many mechanics books, in the chapter on central force motion), the result is

$$t(\theta) = \frac{P}{2\pi} \left[2\tan^{-1} \left(\sqrt{\frac{1-e}{1+e}} \tan\frac{\theta}{2} \right) - \frac{e\sqrt{1-e^2}\sin\theta}{1+e\cos\theta} \right]$$

• This tells you the time t it takes a body to travel in its orbit from periapsis ($\theta = 0$) to any value of θ along the trajectory.

• The fundamental problem here is that this is $t(\theta)$, and generally the most useful quantity is $\theta(t)$. The equation is not invertible, a fact that has spawned a 400 year quest for new and better methods for finding $\theta(t)$. Many analytical methods exist, and many approximations exist, but in the modern era computational technology give us the ability to efficiently solve the orbit equations for the values we want.

• One of the most useful constructs for computational orbit determination is *the Kepler Equation*, which relates the eccentric anomaly ψ to the mean anomaly $\mathcal{M}$.

$$\mathcal{M} = \psi - e\sin\psi$$

• This also is a transcendental equation, but one that is far easier to solve than the equation for $t(\theta)$. Solving this for ψ, one can geometrically show that ψ is related to θ by

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}} \tan\frac{\psi}{2}$$