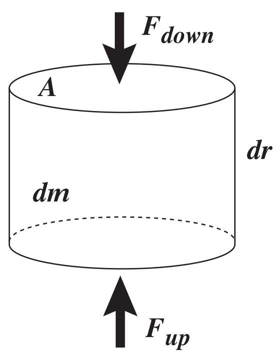
The Interiors of the Stars

Hydrostatic Equilibrium

• Stellar interiors, to a good first approximation, may be understood using basic physics. The fundamental operating assumption here is that the star is in *equilibrium*. There are two primary ways to approaching this — through energy arguments, and through force arguments. This is, you will recall, one of the fundamental lessons from your introductory physics classes — the equivalency of these two approaches. We will use forces for our discussion.

• Consider a small parcel of mass (sometimes called a "fluid element") in the body of the star, cylindrical in shape with a cross sectional area A and depth dr, and a mass $dm = \rho(r)dV$. At fixed radii, the horizontal forces on the element are in equilibrium, so we only consider the differential force on the top of the fluid element compared to the bottom, as shown in the figure below — this is basically a free-body diagram



• Assuming the density is constant ρ across the fluid element, the mass of the element may be written as

$$m \to dm = \rho dV = \rho A dr$$

• At any radius in the star, we assume the forces are *balanced*, otherwise the fluid element would accelerate and the star would fly apart. To convince yourself this is true, consider a fluid element on the surface. The force down is just the force of gravity; the force up is the pressure from beneath (other fluid elements, radiative pressure). These two are balanced, and the star remains a nice sphere.

• We begin by writing Newton's Second Law for the fluid element; we are working under the assumption that on the radial forces (vertical, in the context of our fluid element picture) are important:

$$\sum \vec{F} = m\vec{a} \qquad \to \qquad dm\,\ddot{r} = F_g + F_{P,up} - F_{P,down}$$

• Looking at each of the force terms. The first is just the gravitational force at the radius r of the fluid element. It is

$$F_g = -G\frac{M_r dm}{r^2} = -G\frac{M_r \rho A dr}{r^2}$$

where M_r is the mass of the star interior to the radius of the fluid element. The $(-)$ sign encodes the directionality of the gravitational force, in this case toward the origin of the radial coordinate at the center of the star.

• The other pressure forces act *against each other* (in opposite directions) across the height of the fluid element, dr. Because they act at different depths in the fluid, they are different in magnitude by some amount dF. Remembering that pressure and force are related by $P = F/A$, we may write

$$dF = F_{P,up} - F_{P,down} \qquad \rightarrow \qquad dF = [P(r) - P(r + dr)] \cdot A = -dP \cdot A$$

• Note that $dP = P(r + dr) - P(r)$, giving rise to the $(-)$ sign above. Ultimately this will insure that the sign of *gradients* in the pressure have signs consistent with the definition of positive and negative radial directions.

• Putting this back together in Newton's Second Law, and noting that we have identified the mass of the fluid element to be $m \rightarrow dm = \rho A dr$:

$$dm\, \ddot{r} = \rho A\, dr\, \ddot{r} = -G\frac{M_r \rho A dr}{r^2} - A \cdot dP \qquad \rightarrow \qquad \rho \ddot{r} = -G\frac{M_r \rho}{r^2} - \frac{dP}{dr}$$

• If we assume all the forces are balanced, then the acceleration $\ddot{r} = 0$ and we have an equation for the evolution of the pressure with radius

$$\frac{dP}{dr} = -G\frac{M_r \rho}{r^2}$$

• Remembering that the local acceleration due to gravity is $g(r) = GM_r/r^2$ this becomes

$$\frac{dP}{dr} = -\rho \cdot g(r)$$

which is known as the equation of *hydrostatic equilibrium.* Aficionado's of fluids from your general physics class will recognize that integrating this equation will give the pressure as a function of depth in a constant density fluid: $P(h) = P_o + \rho\, g\, h$. The difference here is that over large radii in the star, $\rho = \rho(r) \neq const$.

EXAMPLE: *We can use this result to make a first estimate of the pressure at the core of the Sun by defining the pressure gradient in one enormous step. Assume the pressure at the surface ($r = 1R_\odot$) is zero: $P_{surf} = 0$.*

The pressure gradient is

$$\frac{dP}{dr} \sim \frac{\Delta P}{\Delta r} = \frac{P_{surf} - P_{core}}{R_{surf} - 0} = \frac{-P_{core}}{R_\odot}$$

This is the left-hand side of the equation of hydrostatic equilibrium. On the right-hand side, we use $g(R_\odot) = G\ M_\odot/R_\odot^2$. What do we use for the density, ρ?

We are treating the star as a single entity rather than integrating over its fine scale structure; in this instance the obvious zeroth order approximation is to take the average density, $\bar{\rho}_\odot = M_\odot/\frac{4}{3}\pi R_\odot^3$. Putting this all together:

$$\frac{dP}{dr} = -G\frac{M_r\rho}{r^2} \qquad \rightarrow \qquad \frac{-P_{core}}{R_\odot} = -G\frac{M_\odot\bar{\rho}_\odot}{R_\odot^2} \qquad \rightarrow \qquad P_{core} = G\frac{M_\odot\bar{\rho}_\odot}{R_\odot}$$

Using $M_\odot = 1.989 \times 10^{30}$ kg and $R_\odot = 6.955 \times 10^8$ m, we find:

$$P_{core} \simeq 2.7 \times 10^{14}\ \mathrm{N/m^2}$$

Stellar Energy

• One of the oldest questions in astrophysics is *how do the stars generate their energy*? We know today that it is nuclear fusion in the core, but this was not understood until the advent of nuclear theory in the early Twentieth Century.

• Another obvious source of energy in the star is the gravitational potential energy. How much gravitational energy is bound up in the star? The potential energy is usually called the *energy of assembly* and is equivalent to the amount of work that is done to bring an ensemble of masses from infinity to build the star.

• Recall that the gravitational potential energy between two masses m_1 and m_2 separated by a distance r is defined as

$$U_E = -G\frac{m_1 m_2}{r}$$

• Note that by this definition, $U_E \rightarrow 0$ as $r \rightarrow \infty$. For all smaller values of r, $U_E < 0$ — the negative value indicates that we have to do work to break the system apart. For this reason, this is sometimes called *binding energy.*

• Gravity is a *conservative force*, which means that the work to move a particle through a gravitational field is simply equal to the change in the potential energy: $W = \Delta U_E$

• Imagine assembling a star by collapsing thin shells of matter from infinity with mass dM and thickness dr. The small amount of work to bring a small shell in is

$$dW = dU_E = dU_{E,f} - dU_{E,i} = -G\frac{M(r)\ dm}{r}$$

where $dU_{E,i} = 0$ at $r = \infty$. Here $M(r)$ is the mass of the star that has already been assembled out to radius r when the thin shell is brought in.

• Re-expressing the mass in the shell in terms of the mass density ρ of the shell

$$dm = \rho \cdot dV = \rho \cdot 4\pi r^2 dr$$

which allows us to write the energy to assemble a thin shell as

$$dU_E = -4\pi G\ M(r)\rho\ r\ dr$$

• Now integrate over the radius of the star to get the total energy of assembly

$$U_E = -4\pi G \int_0^R M(r)\rho\ r\ dr$$

but we don't know how the density ρ varies with radius, and consequently don't know what $M(r)$ is either! In the absence of knowledge, make the most reasonable assumption and justification you can. In this instance, assume the star is a constant average density, from which you can then express $M(r)$:

$$\rho \sim \bar{\rho} = \frac{M}{\frac{4}{3}\pi R^3} \qquad \rightarrow \qquad M(r) \sim \bar{\rho} \cdot V = \frac{M}{\frac{4}{3}\pi R^3} \cdot \frac{4}{3}\pi r^3 = M\left(\frac{r}{R}\right)^3$$

• Using this, our energy integral becomes

$$U_E = -\frac{3GM^2}{R^6}\int_0^R r^4 dr = -\frac{3}{5}\frac{GM^2}{R}$$

• The last step is to employ a result that we will state without proof (for details see §2.4 in BOB), known as the ***virial theorem***. Succinctly stated, the virial theorem is

$$\langle E \rangle = \frac{1}{2}\langle U_E \rangle$$

In words: *for gravitationally bound systems in equilibrium, the total energy is half of the time-averaged potential energy.*

• So employing this without hesitation (using the time-honored code[1] of *"ignorance is bliss"*) we find the total mechanical energy of the star:

$$E \simeq -\frac{3}{10}\frac{GM^2}{R}$$

[1]Remember: codes are more like guidelines, not actual rules.

Kelvin-Helmholtz Timescale

- How could the star use the mechanical energy to power it?
  - ▷ In a non-equilibrium state, there is not enough outward force to oppose the gravitational collapse so the star begins to contract
  - ▷ For the star to contract, gravitational potential energy must be lost; in this case it goes into heating the star at a smaller radius (keeping the star shining)
  - ▷ The star loses energy through radiation at the surface (its luminosity), a rate that proceeds in concert with the stellar temperature.
  - ▷ Energy is radiated away, lowering the temperature of the star, reducing the pressure support from the radiation, and the star contracts a bit. Repeat.
  - ▷ This is a *thermal process* the star uses to convert gravitational energy into luminous energy

• We could estimate the lifetime of a star to be the amount of time it takes the star to completely radiate this energy away. Remember that our physics concept of *power* in astrophysics is *luminosity*:

$$L = \frac{\Delta E}{\Delta t} \qquad \rightarrow \qquad \Delta t = \frac{\Delta E}{L} \qquad \rightarrow \qquad \Delta t = \frac{3}{10}\frac{GM^2}{R\,L}$$

This is called the ***Kelvin-Helmholtz timescale***.

The Big Question

This is the first time we have an astrophysical result that we can use to estimate the age of the Cosmos with. The Universe must be *at least* as old as the stars that inhabit it. Consider the star closest to us: the Sun.

[▶ **EX** ◀] For the Sun: $M_\odot = 1.99 \times 10^{30}$ kg, $R_\odot = 6.96 \times 10^8$ m, $L_\odot = 3.84 \times 10^{26}$ W, then the Kelvin-Helmholtz timescale is:

$$\Delta t_{KH} = 2.967 \times 10^{14}\ \text{s} = 9.40 \times 10^6\ \text{yr} \sim 10^7\ \text{yr}$$

[▶ **Take-away** ◀] We have non-astrophysical means of placing a lower bound on the age of the Cosmos, using radio-carbon dating of rocks on the Earth and Moon, which give ages in excess of 4 billion years. This clearly is not a good estimate of the age of the Cosmos.

But we have still learned something quite valuable: Δt_{KH} is certainly a lower bound on the age of the Sun, but we know it must be wrong since the planets almost certainly formed *after* the Sun. What this really tells us is this: *gravitational potential energy is not the energy source of the stars.*

Energy Transport

• The energy of the star is generated deep down in the core, where the pressure is highest. How does that energy get out? It is *transported*, and that transport has great bearing on the structure of the star

• Early on, you learned there were three energy trnasport mechanisms: *conduction*, *convection* and *radiation*. We will state without proof that conduction is generally not important in stars.

• The guiding principle in energy transport is the star is stable for long periods of time (multi-billions of years). As such, the temperature profile, $T(r)$, must be approximately constant in time. Imagine T at some radius r shrinking or growing compared to the stars around it. The properties of the gas in that stellar layer would change, altering the balance of forces and causing the layer to expand or contract, setting off a chain of responses in other layers of the star.

• If the temperature profile is constant, then the energy entering any shell of the star must be balanced by the energy leaving the shell of the star. With this guiding principle, and assuming radiation transport dominates, we can derive a result for the temperature profile $T(r)$.

• Consider the flux $F(r)$ through a surface at radius r in the star. The Stefan-Boltzmann law tells us that if the surface emits like a blackbody (our default assumption for stars) then

$$F(r) = \sigma T(r)^4$$

• The derivative with respect to temperature T tells us who the flux changes with the temperature at the shell

$$\frac{dF(r)}{dT} = 4\sigma T^3(r) \qquad \rightarrow \qquad dF = 4\sigma T^3(r)\, dT$$

• The star is not transparent to radiation; it has an *opacity*. There are a variety of ways to characterize this, but here it is convenient to define the opacity κ, representing the ability of stellar material to absorb radiation. Simplistically, $\kappa \cdot \rho$ is the fraction of radiation absorbed per unit length. The change in flux dF over some small length dr then is

$$dF = -\kappa(r)\rho(r)F(r)dr$$

where the $(-)$ indicates a *decrease* in flux due to absorption.

• Combining these results we obtain a fundamental equation for the temperature as a function of radius

$$-\kappa(r)\rho(r)F(r)dr = 4\sigma T^3 dT \qquad \rightarrow \qquad F(r) = \frac{L}{4\pi r^2}$$

• Solving this for the luminosity L tells us how the rate of energy flow depends on the temperature gradient in the star

$$L(r) = -\left[\frac{16\pi\sigma r^2 T(r)^3}{\kappa(r)\rho(r)}\right]\frac{dT}{dr}$$

• In the usual way of theoretical physics, this must look like we are going around in circles! Now we have the unknown luminosity to deal with! And the density! Our goal always is to *reduce* to the minimum number of things we have to guess at. Let's see if we can get rid of the luminosity!

• Supposs we define ϵ to be the total amount of energy per unit mass generated by the star[2]. Then a small parcel dm in the star contributes to the stars luminosity

$$dL = \epsilon\, dm$$

• Expressing the mass element in terms of the density and volume of the shell

$$dm = \rho(r)\cdot 4\pi r^2 dr \qquad \rightarrow \qquad dL = \epsilon\rho(r)\cdot 4\pi r^2 dr \qquad \rightarrow \qquad \frac{dL}{dr} = 4\pi r^2\rho(r)\epsilon$$

• So now I have the uknown density function $\rho(r)$ and this widget ϵ which is also radially dependent: $\epsilon = \epsilon(r)$. Am I getting nowhere? Actually, we can say something about ϵ from nuclear physics.

Aside: How do stars hold together with all this energy?

• Why doesn't the enormous pressure from the generation of all this radiant energy blow a star apart?

• The star is stable because it has a *negative heat capacity.* What does that mean?

• The heat capacity is the energy required to raise the temperature of a material by a given amount. Consider the energetics of the gas the star is made from. $E = K_E + U_E$, where K_E is the thermal energy of the particles. Using the virial theorem:

$$E = K_E + U_E \underbrace{=}_{virial} \frac{1}{2}U_E \qquad \rightarrow \qquad K_E = -\frac{1}{2}U_E = -E$$

• Now, suppose I add energy, making E *increase* — E becomes *less negative.* That means K_E becomes *less positive*, or a *decrease in thermal energy*, the gas cools!

• So if the star produces too much energy for its equilibrium configuration it expands and cools to adjust.

[2]You may remember from thermodynamics that dividing any quantity by the mass garners it the label *specific*, so ϵ could be called the *specific energy.*